

## Some Category-Theoretic Ideas in Algebra (A Too-Brief Tour of Algebraic Structure, Monads, and the Adjoint Tower)

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In recent years, categorists have come up with some very interesting ways of looking at algebraic constructions and algebraic objects. But most of what they write on this is technical and aimed at other categorists. I shall sketch some of these ideas here, emphasizing concrete examples, for the algebraist with a reasonable foundation in category theory (familiarity with adjoint functors and colimits). The unifying thread of the article will be the problem: What algebraic structure can be put on the values of a given set-valued functor?

**1. Coalgebras, and representable functors [1]—review.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be varieties of algebras. ( $\mathcal{A}$  may be, more generally, any category with colimits.) It is known that a functor  $V: \mathcal{A} \rightarrow \mathcal{B}$  has a left adjoint if and only if at the set level it is representable; that is, if and only if, letting  $U: \mathcal{B} \rightarrow \mathcal{C}^{\text{ens}}$  denote the underlying-set functor of  $\mathcal{B}$ , one has  $U \circ V \cong \text{Hom}(R, \text{—})$  for some object  $R$  of  $\mathcal{B}$ :

$$(1) \quad \begin{array}{ccc} & \mathcal{B} & \\ V \nearrow & \downarrow U & \\ \mathcal{A} & \xrightarrow{\quad} & \mathcal{C}^{\text{ens}} \\ & \text{Hom}(R, \text{—}) & \end{array}$$

In this situation, the structures of algebra  $V(A) \in \text{Ob}(\mathcal{B})$  on the sets  $\text{Hom}(R, A)$  arise from a  $\mathcal{B}$ -coalgebra structure on the representing object  $R$  in  $\mathcal{A}$ .

**EXAMPLE.** The functor  $\text{GL}_n: \mathcal{Ring} \rightarrow \mathcal{Group}$  has a left adjoint, because  $U \circ \text{GL}_n$  is represented by the ring  $R$  presented by  $2n^2$  generators  $x_{ij}, y_{ij}$  ( $i, j \leq n$ ) and the  $2n^2$  relations comprising the matrix equations  $((x_{ij}))((y_{ij})) = ((y_{ij}))((x_{ij})) = I_n$ , i.e.,

the ring with a *universal* invertible  $n \times n$  matrix,  $x = ((x_{ij}))$ . To study the multiplication of  $GL_n$  take the ring with *two* universal invertible  $n \times n$  matrices, namely the coproduct of two copies of  $R$ ,  $R' \coprod R''$ , and call these two matrices  $x', x'' \in GL_n(R' \coprod R'')$ ; they correspond to the two coprojection maps,  $R \rightarrow R' \coprod R''$ . Form their product  $x'x'' \in GL_n(R' \coprod R'')$ , and represent it by a homomorphism  $m: R \rightarrow R' \coprod R''$ . The homomorphism  $m$  now “encodes” the multiplication of  $GL_n$ , just as the object  $R$  “encodes” the construction of  $GL_n$  as a set: Given any elements  $a, b \in GL_n(A) = \text{Hom}(R, A)$  (any ring  $A$ ) one gets their product in  $GL_n(A) = \text{Hom}(R, A)$  as the composition:

$$R \xrightarrow{m} R' \coprod R'' \xrightarrow{(a, b)} A.$$

In the same way, the matrix-inverse operation of  $GL_n$  corresponds to a map  $i: R \rightarrow R$  (namely,  $x_{ij} \mapsto y_{ij}$ ,  $y_{ij} \mapsto x_{ij}$ ); and the 0-ary operation giving the identity matrix  $I_n \in GL_n(A)$  corresponds to a map of  $R$  into its 0-fold coproduct with itself,  $\coprod_{\emptyset} R$ , which is simply the initial object  $\mathbf{Z}$  of  $\mathcal{R}ing$  (namely,  $x_{ij}, y_{ij} \mapsto \delta_{ij} \in \mathbf{Z}$ ). These maps, called *comultiplication*, *coinverse* and *counit*, comprise a structure of *cogroup* on the object  $R$  of  $\mathcal{R}ing$ .

For a very interesting exposition of coalgebras and related constructions, see P. Freyd [1]. Cf. also [2, §III.6] and [4].

(Warning to the ring-theorist: Do not confuse this use of the term “cogroup,” and, more generally, of “coalgebra” with the deceptively similar meaning of the latter term in the theory of Hopf algebras! The relation between these concepts is discussed in [4, §8].)

**2. Turnabout is fair play.** Let us now reverse our viewpoint. Let  $\mathcal{A}$  be a category with coproducts, and  $R$  be an object of  $\mathcal{A}$ . Suppose we form the representable functor  $\text{Hom}(R, -): \mathcal{A} \rightarrow \mathcal{Ens}$ , and ask: What algebraic structure can we put on this functor? That is, what is the richest category of algebras  $\mathcal{B}$  such that we can factor  $\text{Hom}(R, -)$  through the forgetful functor  $U: \mathcal{B} \rightarrow \mathcal{Ens}$  as in (1)?

The remarks of the preceding section contain the answer: The  $n$ -ary *operations* we can put on  $\mathcal{B}$  correspond precisely to the  $n$ -ary *cooperations* possessed by  $R$  in  $\mathcal{A}$ , i.e., to the set of all maps  $R \rightarrow R \coprod \cdots \coprod R$ . In general this will give a very big and unwieldy set of operations, but there may be some convenient subset which *generates* the rest.

The identities of  $\mathcal{B}$  will come from “coidentities” of these co-operations of  $R$ .

EXAMPLE. What algebraic structure can we put, in a functorial manner, on the set of elements of exponent 2 in a group  $G$ ?

The functor  $G \mapsto \{x \in |G| \mid x^2 = e\}$  is represented by the object  $\mathbf{Z}_2$  of  $\mathcal{Group}$ . A description of all maps  $\mathbf{Z}_2 \rightarrow \mathbf{Z}_2 \coprod \cdots \coprod \mathbf{Z}_2$ , i.e., of all elements of exponent 2 in the group with presentation  $\langle x_1, \dots, x_n \mid x_1^2 = \cdots = x_n^2 = e \rangle$ , may be obtained from classical results on the structure of coproducts of groups. (N.B. Not by “general nonsense”!) They are (as elements) precisely  $e$ , and all conjugates of the generators  $x_1, \dots, x_n$ . From this it is not hard to deduce that the operations we get on  $\text{Hom}(\mathbf{Z}_2, -)$  are generated by the 0-ary operation  $e$  (induced by the trivial map

$\mathbf{Z}_2 \rightarrow \coprod_{\emptyset} \mathbf{Z}_2 = \{e\}$ ) and the binary operation of conjugation,  $(x, y) \mapsto x^y = y^{-1}xy$  (induced by the map  $\mathbf{Z}_2 = \langle t \mid t^2 = e \rangle \rightarrow \mathbf{Z}_2 \coprod \mathbf{Z}_2 = \langle x, y \mid x^2 = y^2 = e \rangle$  taking  $t$  to  $x^y$ ).

A group-theoretic analysis of when iterated conjugates of generators in groups  $\mathbf{Z}_2 \coprod \cdots \coprod \mathbf{Z}_2$  coincide leads to the result that all identities satisfied by these two operations follow from the following five:

$$\begin{aligned} e^x &= e, & x^x &= x, & x^{(y^z)} &= ((x^z)y)^x, \\ x^e &= x, & (xy)^y &= x, \end{aligned}$$

Hence let us call an algebra of type  $(0, 2)$  (i.e., a set with one zero-ary and one binary operation) satisfying these five identities an “involution algebra”. Then the variety  $\mathcal{I}_{ns}$  of all involution algebras is the richest variety “ $\mathcal{B}$ ” through which  $\text{Hom}(\mathbf{Z}_2, -): \text{Group} \rightarrow \mathcal{E}_{ns}$  can be factored as in (1). This factorization corresponds to a structure of *involution coalgebra* on the representing object  $\mathbf{Z}_2$  in  $\text{Group}$ .

For other examples see [20].

**3. Interpretation in terms of Lawvere's algebraic theories and algebraic structure.** W. Lawvere introduced in his thesis [3] the idea of describing any variety  $\mathcal{A}$  of algebras as the category  $\mathcal{E}^{ns\theta}$  of all finite-direct-product preserving functors from a certain category  $\theta$ , called the “theory” of  $\mathcal{A}$ , into the category  $\mathcal{E}_{ns}$  of sets. The category  $\theta$  consists of an object  $1$ , and finite products  $1 \times \cdots \times 1$ , and has for morphisms, in addition to maps constructible from projections, certain other maps corresponding to the *operations* of  $\mathcal{A}$ , with relations among their compositions corresponding to the *identities* of  $\mathcal{A}$ .

Actually, Lawvere defines the theory to be the opposite category,  $T$ , to the category  $\theta$  I have described, so that he writes  $\mathcal{A} = \mathcal{E}^{nsT^{\text{op}}}$ . This  $T$  is a little less natural to picture than  $\theta$ , but has a formal advantage: The category *freely* generated by one object  $1$  under finite *coproducts* is (up to equivalence) the full subcategory of  $\mathcal{E}_{ns}$  with object-set  $\omega = \{0, 1, 2, \dots\}$ ; so Lawvere's algebraic theories  $T$  are precisely the coproduct-preserving and object-set-preserving extensions of that category.

The “theory”  $\theta$  (respectively  $T$ ) of a variety  $\mathcal{A}$  can be looked at as the category with a universal  $\mathcal{A}$ -algebra object (respectively co- $\mathcal{A}$ -algebra object)  $1$ . Thus, in the category of all categories-with-finite-(co)products, and functors respecting these,  $\theta$  (resp.  $T$ ) represents the construction associating to a category  $\mathcal{C}$  the category of all  $\mathcal{A}$ -(co)algebras in  $\mathcal{C}$ :

$$\mathcal{A}\text{-alg}(\mathcal{C}) \cong \mathcal{C}^{\theta}, \text{ respectively } \mathcal{A}\text{-coalg}(\mathcal{C}) \cong \mathcal{C}^T.$$

One can also show that  $T$  is isomorphic to the full subcategory of  $\mathcal{A}$  having for object-set the set of free algebras  $\{F(0), F(1), \dots\}$ .

One may now check that the variety  $\mathcal{B}$  we associated to any representable functor  $\text{Hom}(R, -): \mathcal{A} \rightarrow \mathcal{E}_{ns}$  in the preceding section is described in Lawvere's terms as  $\mathcal{E}^{nsT^{\text{op}}}$  where  $T$  is the full subcategory of  $\mathcal{A}$  with objects  $\coprod_n R$  ( $n = 0, 1, 2, \dots$ )!

Lawvere looked, too, at the question of what algebraic structure can be put on a functor  $V: \mathcal{A} \rightarrow \mathcal{E}_{ns}$ , or more generally,  $\mathcal{A} \rightarrow \mathcal{C}$  where  $\mathcal{C}$  is any category with finite direct products. He observes that a functorial  $n$ -ary operation on the  $V(A)$ 's just

means a morphism (natural transformation) of functors,  $V^n \rightarrow V$ . The full subcategory of the functor-category  $\mathcal{C}^{\mathcal{A}}$ , with object-set  $\{V^0, V^1, V^2, \dots\}$ , will form an algebraic theory  $\theta_V$  (unless  $V$  is trivial), which defines as above a variety  $\mathcal{B}$  such that the values of  $V$  can be regarded as  $\mathcal{B}$ -objects in  $\mathcal{C}$ . Lawvere calls the theory  $\theta_V$ , or rather its opposite,  $T_V$ , the “algebraic structure” of  $V$ .

If  $V: \mathcal{A} \rightarrow \mathcal{C}^{\mathcal{A}}$  is a representable functor, say  $V = \text{Hom}(R, -)$ , we see from the Yoneda lemma that this category  $T_V$  will be isomorphic to the subcategory of  $\mathcal{A}$  with objects  $\{\coprod_n R \mid n = 0, 1, \dots\}$  which we used to define the  $\mathcal{B}$  of the preceding section. Thus,  $\mathcal{B} = \mathcal{C}^{\mathcal{A}^{T_V}}$ , so the algebraic structure on  $V$  that we determined in the preceding section is indeed the algebraic structure of  $V$  in Lawvere’s sense.

However, we shall see in §5 that there are in general also “higher” types of algebraic structure to be found in a functor  $V$ !

If  $T$  is an algebraic theory, Lawvere calls the associated variety  $\mathcal{C}^{nsT}$  the “semantics” of  $T$ . Thinking of  $\mathcal{C}^{nsT}$  as a category given with a (forgetful) functor  $U$  to  $\mathcal{C}^{ns}$ , i.e., an object of  $(\text{Cat} / \mathcal{C}^{ns})$ , the universality of  $\mathcal{B} = \mathcal{C}^{nsT_V}$  as a variety of algebras through which to factor  $V: \mathcal{A} \rightarrow \mathcal{C}^{ns}$  (an arbitrary member of  $(\text{Cat} / \mathcal{C}^{ns})$ ) is expressed by Lawvere’s celebrated result, “Structure is adjoint to semantics”;

$$\begin{array}{ccc} & \text{semantics} & \\ \text{Algebraic Theories} & \longleftrightarrow & (\text{Cat} / \mathcal{C}^{ns}) \\ & \text{structure} & \end{array}$$

**4. Monads.** (For more details see [2, Chapter VI], [6, Introduction], [14, Chapter 21].) We consider again a pair of adjoint functors,

$$\begin{array}{ccc} & V & \\ \mathcal{A} & \xleftrightarrow{F} & \mathcal{C}, \text{ with unit } \eta: 1_{\mathcal{C}} \rightarrow VF, \text{ counit } \varepsilon: FV \rightarrow 1_{\mathcal{A}}. & \\ & F & \end{array}$$

If we forget the category  $\mathcal{A}$ , how much information about this adjunction can we “remember” in terms of the category  $\mathcal{C}$ ?

The composite  $VF$  is an endofunctor  $M: \mathcal{C} \rightarrow \mathcal{C}$ , and the unit  $\eta$  is a morphism  $1_{\mathcal{C}} \rightarrow M$  so these are already expressible in terms of  $\mathcal{C}$ .

The counit  $\varepsilon: FV \rightarrow 1_{\mathcal{A}}$  cannot itself be described in  $\mathcal{C}$ , but  $V\varepsilon F$  will be a morphism  $\mu: MM \rightarrow M$  of endofunctors of  $\mathcal{C}$ .

Writing this “ $\mathcal{C}$ -data” on our adjunction as a 3-tuple  $\mathcal{M} = (M, \eta, \mu)$ , one finds that  $\mathcal{M}$  will satisfy the identities indicated by the commuting diagrams:

$$(2) \quad \begin{array}{ccc} M = 1_{\mathcal{C}}M & \xrightarrow{\eta \cdot 1_M} & MM \\ \parallel & & \downarrow \mu \\ M1_{\mathcal{C}} & & 1_M \\ \downarrow 1_M \cdot \eta & & \downarrow \mu \\ MM & \xrightarrow{\mu} & M \end{array} \quad \begin{array}{ccc} MMM & \xrightarrow{\mu \cdot 1_M} & MM \\ \downarrow 1_M \cdot \mu & & \downarrow \mu \\ MM & \xrightarrow{\mu} & M \end{array}$$

An endofunctor  $M$  of a category  $\mathcal{C}$  given with morphisms  $\eta$  and  $\mu$  satisfying these identities is called a *monad* (because of the parallel with operations  $e: 1 \rightarrow X$ ,  $m: X \times X \rightarrow X$ , and the corresponding identities, defining a *monoid*  $(X, e, m)$ ! Another common term for monad is *triple*.)

As an example, consider the *underlying-set* functor  $V: \mathbf{Group} \rightarrow \mathcal{E}^{\text{ns}}$ , its left adjoint  $F$ , and the resulting monad  $\mathcal{M} = (M, \eta, \mu)$ . The functor  $M = VF: \mathcal{E}^{\text{ns}} \rightarrow \mathcal{E}^{\text{ns}}$  takes a set  $S$  to the set of elements of the free group on  $S$ , which can be thought of as the set of all “abstract group-theoretic combinations of the elements of  $S$ ”. The description of  $\eta$  is clear. The morphism  $\mu$  corresponds to the observation that an “abstract group-theoretic combination of abstract group-theoretic combinations of elements of  $S$ ” can be “reduced”, by composition of operations, to a single abstract combination of elements of  $S$ .

From this monad  $\mathcal{M}$  on  $\mathcal{E}^{\text{ns}}$ , can we reconstruct the original adjunction  $\mathbf{Group} \rightleftarrows \mathcal{E}^{\text{ns}}$  and in particular recover the category  $\mathbf{Group}$ ? The answer is both a resounding “Yes!” and a definitive “No!”

To see the “yes”, note that a group can be described as a set  $S$ , with “a way of evaluating within  $S$  all abstract group-theoretic combinations of its elements”, i.e., a map  $\alpha: M(S) \rightarrow S$ . One finds that the conditions  $\alpha$  must satisfy for such a formal evaluation procedure really to be a group structure are the commutativity of the diagrams:

$$(3) \quad \begin{array}{ccc} S & \xrightarrow{\eta(S)} & M(S) \\ & \searrow 1_S & \downarrow \alpha \\ & S & \end{array} \quad \begin{array}{ccc} M(M(S)) & \xrightarrow{1_M \cdot \alpha} & M(S) \\ \downarrow \mu(S) & & \downarrow \alpha \\ M(S) & \xrightarrow{\alpha} & S \end{array}$$

To see the “no,” let  $\text{tf.Gp}$  denote the category of torsion-free groups, and note that the forgetful functor  $\text{tf.Gp} \rightarrow \mathcal{E}^{\text{ns}}$  also has the free group construction as left adjoint. This adjunction clearly yields the same monad on  $\mathcal{E}^{\text{ns}}$  that we have just considered; so the monad  $\mathcal{M}$  does not uniquely determine the adjoint pair, and in particular, the other category of that pair.

The general situation is this: Given a monad  $\mathcal{M} = (M, \eta, \mu)$  on a category  $\mathcal{C}$ , we may form a category whose objects are pairs  $(S, \alpha)$ ,  $S$  an object of  $\mathcal{C}$ ,  $\alpha$  a morphism  $M(S) \rightarrow S$  satisfying (3), and whose morphisms are object-maps making the obvious square commute. This is called the category of “algebras with respect to  $\mathcal{M}$ ” and denoted  $\mathcal{C}^{\mathcal{M}}$ , and we get an adjunction

$$(4) \quad \begin{array}{ccc} (S, \alpha) \mapsto S & & \\ \mathcal{C}^{\mathcal{M}} & \rightleftarrows & \mathcal{C}, \\ (M(S), \mu(S)) \mapsto S & & \end{array}$$

which is in an appropriate sense (§7 below) *universal* among adjoint pairs inducing  $\mathcal{M}$  on  $\mathcal{C}$ . It is *not* the *unique* pair inducing  $\mathcal{M}$ ; nonetheless many of the most important adjoint pairs are related to their monads in this manner.

In particular, any variety  $\mathcal{A}$  of algebras is equivalent to  $\mathcal{E}^{\text{ns}}^{\mathcal{M}}$ , where  $\mathcal{M}$  is the monad on  $\mathcal{E}^{\text{ns}}$  induced by the underlying/free adjunction  $\mathcal{A} \rightleftarrows \mathcal{E}^{\text{ns}}$ . In fact, there is a 1-1 correspondence between monads  $\mathcal{M}$  on  $\mathcal{E}^{\text{ns}}$  and varieties of algebras! Given a monad  $\mathcal{M}$ ,  $\mathcal{E}^{\text{ns}}^{\mathcal{M}}$  can be made a variety whose  $n$ -ary operations are the elements of  $M(n)$ , for each  $n$ . (Again, of course, in particular cases one may have much smaller

generating sets of operations.) The identities, i.e., the rules for composing operations, are given by  $\mu$ .

Actually, some set-theoretic qualifications are needed here. A precise statement is that monads on  $\mathcal{E}^{\text{ns}}$  correspond to varieties of algebras which *may* have infinitary operations ( $n$  must range through all cardinals, in the preceding statement) and whose operations may even form a proper class; but such that there are only a set of distinct derived operations of each arity. Varieties of “finitary” algebras correspond to those monads  $\mathcal{M}$  such that for all  $S \in \mathcal{E}^{\text{ns}}$ ,  $M(S) \cong \text{colim}_{S_0 \text{ finite}, \subseteq S} M(S_0)$ .

We can now give yet another view of our construction of the “structure” on a representable functor  $V = \text{Hom}(R, \text{—})$ :  $\mathcal{A} \rightarrow \mathcal{E}^{\text{ns}}$  ( $\mathcal{A}$  a category with coproducts). The auxiliary variety  $\mathcal{B}$  through which we factored  $V$  was precisely  $\mathcal{E}^{\text{ns}}\mathcal{M}$ , where  $\mathcal{M}$  is the monad on  $\mathcal{E}^{\text{ns}}$  induced by  $V$  and its left adjoint  $F$ .

$$\begin{array}{ccc} & \mathcal{E}^{\text{ns}}\mathcal{M} = \mathcal{B} & \\ V_2 \nearrow & & \downarrow \\ \mathcal{A} & \xleftarrow{F} & \mathcal{E}^{\text{ns}} \end{array}$$

(This is imprecise because we only considered algebraic structure based on finitary operations in preceding sections. We may correct this by (a) allowing infinitary operations in our earlier discussions; or (b) replacing  $\mathcal{M}$  by the submonad  $\mathcal{M}_{\text{fin}}$ , where  $\mathcal{M}_{\text{fin}}(S) = \text{colim } M(S_0)$ , thus discarding infinitary operations; or (c) if  $\mathcal{A}$  is a variety, and the object  $R$  representing  $V$  is finitely generated, as was true for  $\mathbb{Z}_2$  in groups, by noting that then  $\mathcal{M} = \mathcal{M}_{\text{fin}}$ , so in this case there is no problem.)

**5. Higher structure.** We have seen that the categories of algebras with respect to monads on  $\mathcal{E}^{\text{ns}}$  are varieties of algebras in the traditional sense. What, then, will we get if we *start* with a variety  $\mathcal{C}$  of algebras and a monad  $\mathcal{M}$  on  $\mathcal{C}$ , and form the category of algebras  $\mathcal{C}^{\text{II}}$ ?

It turns out that the objects of  $\mathcal{C}^{\text{II}}$  can be described as sets endowed with, *in addition* to the operations of  $\mathcal{C}$ , certain *partial* operations, and subject, in addition to the identities of  $\mathcal{C}$ , to certain “partial identities”. Explicitly, if  $A \in \text{Ob}(\mathcal{C})$  is an object definable by generators  $X_1, \dots, X_n$  and a system of relations  $r(X)$ , then each element of  $M(A)$  induces a partial operation on the objects  $B \in \mathcal{C}^{\text{II}}$ , whose domain is the set of all  $n$ -tuples  $(x_1, \dots, x_n) \in |B|^n$  satisfying  $r(x)$ . (Thus the domains of these “second-stage” operations are defined with the help of the “first-stage” operations, those of  $\mathcal{C}$ .) Likewise, the map  $\mu: M(M(A)) \rightarrow A$  gives identities in these partial (and total) operations which must be satisfied by all  $n$ -tuples satisfying  $r$ .

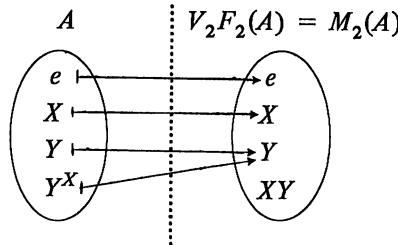
Again, for illustration consider the functor  $V_1 = \text{Hom}(\mathbb{Z}_2, \text{—}): \text{Group} \rightarrow \mathcal{E}^{\text{ns}}$ , and its lifting  $V_2: \text{Group} \rightarrow \mathcal{I}_{\text{Inv}}$  ( $\mathcal{I}_{\text{Inv}}$  = the variety of involution algebras).  $V_2$  has a left adjoint  $F_2$ , so this adjoint pair will induce a monad  $\mathcal{M}_2$  on  $\mathcal{I}_{\text{Inv}}$  . . . .

What does this mean concretely? An involution algebra  $A$  gives in a natural manner generators and relations for a certain group  $F_2(A)$ —exponent 2-generators, and conjugacy relations among these. This group can be characterized as having a

universal map of involution algebras  $A \xrightarrow{\eta_2(A)} V_2 F_2(A)$ . Now if our definition of “involution algebra” were a really complete picture of the structure of the sets of elements of exponent 2 in groups, we would expect the maps  $\eta_2(A)$  to be isomorphisms. (Or if you don’t buy that, let us just say it is natural, in studying elements of exponent 2 in groups, to ask whether this map will be an isomorphism.)

But  $\eta_2$  is in general neither injective nor surjective. For example, let  $A$  be the involution algebra defined by two generators  $X, Y$  and one relation  $XY = X$ . One finds that  $A$  has underlying set  $\{e, X, Y, Y^X\}$ .  $F_2(A)$  will be the group on generators  $X, Y$  and relations  $X^2 = Y^2 = e, Y^{-1}XY = X$ .

The latter relation says that  $X$  and  $Y$  commute, so  $F_2(A)$  is the fours-group, with underlying set  $\{e, X, Y, XY\}$ , and all its elements have exponent 2. Hence the map  $\eta_2(A)$  takes the form

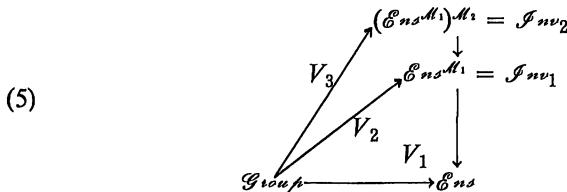


which is neither surjective nor injective. The “new” element  $XY$  in  $M_2(A)$  leads to a partial binary operation on elements of exponent 2 in a group, associating to every pair  $(x, y)$  such that  $x^y = x$  the element  $xy$  (which will have exponent 2 precisely because  $x$  and  $y$  commute). This operation is not definable in terms of conjugation.

The collapse of  $Y$  and  $Y^X$  in  $M_2(A)$  likewise yields the “partial identity” (Horn sentence) holding in the involution algebra of any group, but not following from the full identities of involution algebras:

$$(\forall x, y) \ x^y = x \Rightarrow y^x = y.$$

If we gather together the partial operations and partial identities arising from the maps  $\eta_2(A)$  for all  $A \in \text{Ob}(\mathcal{I}_{\text{ns}})$ , and add these to our earlier list of operations and identities, we get axioms for what we may call a “second order involution algebra.” In fact, the category  $\mathcal{I}_{\text{nv2}}$  of second order involution algebras is precisely  $\mathcal{I}_{\text{nv1}}^{\mathcal{M}_1} = (\mathcal{C}_{\text{ns}}^{\mathcal{M}_1})^{\mathcal{M}_1}$ . (I am now using the subscript 1 for “first order” involution algebras, i.e., what I previously just called involution algebras.) Since we get this additional structure on the objects  $V_2(G)$  for any group  $G$ , we get a second factorization:



All this applies, mutatis mutandis to any representable set-valued functor  $V$  on a cocomplete category  $\mathcal{A}$ , in particular, on any variety  $\mathcal{A}$  of algebras.

One can continue to iterate this process. At each step one obtains operations and identities whose *domains* are given by systems of equations in the previously constructed operations. The resulting diagram ((5) extended) is called the *adjoint tower* induced by the original functor  $V$ . The construction is due to Applegate and Tierney; cf. [12].

Note that the problem of explicitly studying these classes of algebras is not one of general nonsense but, for instance, in the case  $V_1 = \text{Hom}(\mathbf{Z}_2, -)$ , real group theory. I do not know, for example, whether  $\mathcal{I}_{\text{nnv}_2}$  can be presented by finitely many partial operations and identities. I do not know whether at the *next* step one would find  $\eta_3: A \rightarrow V_3 F_3(A)$  always to be an isomorphism—in which case the tower would become constant after that point, and  $\mathcal{I}_{\text{nnv}_2}$  would be equivalent to a full coreflective subcategory of  $\mathcal{G}_{\text{roup}}$ , via  $F_3$  and  $V_3$ —or not. Something positive that one *can* say in this case, because  $\mathbf{Z}_2$  is finitely presented as a group, is that on the category  $\mathcal{I}_{\text{nnv}_\omega} =_{\text{def}} \lim_i ((\cdots \mathcal{E}^{\text{ns}M_i} \cdots)^M)$  (the natural “ $\omega$ th step of the adjoint tower”)—sets with all the structure one gets at the finite steps,  $\mathcal{I}_{\text{nnv}_i}$ , the maps  $\eta_\omega(A)$  will indeed be isomorphisms, and so  $\mathcal{I}_{\text{nnv}_\omega}$  is equivalent to a full coreflective subcategory of  $\mathcal{G}_{\text{roup}}$ . On the other hand, there *are* examples of adjoint towers of arbitrary transfinite height.

Let us note how Lawvere's approach to algebraic theories can be extended to these higher sorts of algebras. Given a category  $\mathcal{A}$  with colimits, and an object  $R$  in  $\mathcal{A}$ , let  $\mathcal{B}_i$  denote the category  $((\cdots \mathcal{E}^{\text{ns}M_i} \cdots)^M)$  arising at the  $i$ th level of the adjoint tower induced by  $\text{Hom}(R, -)$ . We recall that  $\mathcal{B}_1$ , a variety, may be identified with the category of all product-respecting functors  $T_1^{\text{op}} \rightarrow \mathcal{E}^{\text{ns}}$ , where  $T_1$  is the full subcategory of  $\mathcal{A}$  having for object-set the coproducts of copies of  $R$  (in other words, the object-image of the adjoint  $F_1: \mathcal{E}^{\text{ns}} \rightarrow \mathcal{A}$ ). Again, if as in §2 we are interested only in finitary operations, we just use  $\{F_1(n) \mid n < \omega\}$ ; and in fact if we make no such restriction at all there are set-theoretic worries; but we shall skip over these here). Likewise, to describe  $\mathcal{B}_2$  let  $T_2$  denote the full subcategory of all colimits in  $\mathcal{A}$  of objects of  $T_1$ ; then we find that  $\mathcal{B}_2$  is equivalent to the category  $\mathcal{E}^{\text{ns}T_2^{\text{op}}}$ , where by this we mean all functors  $T_2^{\text{op}} \rightarrow \mathcal{E}^{\text{ns}}$  respecting these (co)products and (co)limits. The object-set of  $T_2 \subseteq \mathcal{A}$  can also be described as the image of  $F_2: \mathcal{B}_1 \rightarrow \mathcal{A}$ .

For instance, in our  $\mathbf{Z}_2$  example,  $T_2$  contains not only the groups  $\mathbf{Z}_2 \coprod \dots \coprod \mathbf{Z}_2$  but also the difference-cokernel  $H$ :

$$\mathbf{Z}_2 \xrightarrow[t \mapsto x]{ } \mathbf{Z}_2 \coprod \mathbf{Z}_2 \xrightarrow{\quad} H \text{ (the fours-group).}$$

Hence  $T_2$  contains the map

$$\mathbf{Z}_2 \xrightarrow[t \mapsto xy]{ } H$$

which induces in  $\mathcal{B}_2$  the partial operation we discovered.

Note that given the “theory” of  $\mathcal{I}_{\text{nnv}_2}$ , either in the classical sense of a list of partial operations and identities, or in this Lawverian form, we can speak of “ $\mathcal{I}_{\text{nnv}_2}$

“objects” in a general category. In particular,  $\mathbf{Z}_2$  is now a co- $\mathcal{I}_{\text{new}_2}$ -object in the category  $\text{Group}$ . (And in our general context, with  $V_1 = \text{Hom}(R, -)$ ,  $R$  will be a co- $\mathcal{B}_n$ -object of  $\mathcal{A}$  for all  $n$ .)

Lawvere’s concept of the structure of a functor likewise generalizes naturally to this higher structure: In the formulation of that concept, where Lawvere took the full subcategory of the functor-category  $\mathcal{C}^{\mathcal{A}}$  with object-set  $\{V^n \mid n = 0, 1, \dots\}$ , one merely considers instead the full closure under (finite) limits of  $\{V\}$  in  $\mathcal{C}^{\mathcal{A}}$ . For instance, our partial multiplication operation on  $\text{Hom}(\mathbf{Z}_2, -)$  now assumes the form of a morphism of functors  $W \rightarrow V$ , where  $W \in \mathcal{C}^{\text{ns} \times \mathcal{G}_{\text{new}}}$  is the difference-kernel functor

$$\begin{array}{ccc} & (x, y) \mapsto x & \\ W \longrightarrow V^2 & \xrightarrow{\quad} & V \\ & (x, y) \mapsto xy & \end{array}$$

**6. Various examples and observations.** The reader will find it instructive (and not too difficult) to describe completely the adjoint towers associated with the following functors: the underlying set functor of an arbitrary variety of algebras; the underlying set functor of the category (quasi-variety) of torsion-free groups; (example of A. Stone); the functor  $\text{Hom}(2, -): \mathcal{Poset} \rightarrow \mathcal{C}^{\text{ns}}$ , where  $2 \in \text{Ob}(\mathcal{Poset})$  is the chain of length 1, the same example with  $\mathcal{Cat}$ , the category of small categories, in place of  $\mathcal{Poset}$ . (For 2 is also a category. Cf. [13, Q13].)

(From the torsion-free groups example one can generalize to get a characterization of quasi-varieties and semivarieties as certain categories of the form  $(\mathcal{C}^{\text{ns} \times \mathcal{M}})^{\mathcal{M}}$ .)

Suppose  $A, B, C$  are associative rings with 1, given with maps  $A \rightarrow B, A \rightarrow C$ . Then for any right  $B$ -module  $M$  and right  $C$ -module  $N$ , we can form  $\text{Hom}_A(M, N)$ , getting a functor  $V: (\text{Mod } B)^{\text{op}} \times (\text{Mod } C) \rightarrow \mathcal{C}^{\text{ns}}$  (not representable in the sense we have been considering). Through universal tricks, one can determine the structure of  $V$  in the sense of Lawvere: It is that of a  $B \odot_A C$ -module, where  $B \odot_A C = \{x \in B \otimes_A C \mid \forall a \in A, ax = xa\}$ , made a ring in a certain natural manner discovered by M. Sweedler. There can also be higher structure.

If we start with a family of objects  $(R_i)_{i \in I}$  in a category  $\mathcal{A}$ , they induce a functor  $(\text{Hom}(R_i, -))_{i \in I}: \mathcal{A} \rightarrow \mathcal{C}^{\text{ns} I}$ , which we can examine for structure of *many-sorted algebra*. In the category  $\mathcal{HTop}^*$  of pointed topological spaces with homotopy classes of maps, the family of spheres,  $(S^i)_{i \in N}$  induces the functor  $\pi_*: \mathcal{HTop}^* \rightarrow \mathcal{C}^{\text{ns} N}$ . The structure of this functor includes not only the group structures of each homotopy group  $\pi_n$ , but also operations between different degrees, e.g., the “Whitehead products”  $\pi_m \times \pi_n \rightarrow \pi_{m+n-1}$ , induced by maps  $S^{m+n-1} \rightarrow S^m \coprod S^n$ .

For applications of ideas related to those of this article to the foundations of algebraic geometry and differential geometry, see [6, pp. 146–244], [9], [10], [11]; for applications to measure theory, see [17].

By duality, one can apply the ideas we have discussed to representable contravariant functors. For instance, the (finitary) structure on the functor  $\text{Hom}(-, 2): \mathcal{Poset} \rightarrow \mathcal{C}^{\text{ns}}$  turns out to be precisely that of *distributive lattices*. (Exercise. Examine similarly  $\text{Hom}(-, 2): \mathcal{C}^{\text{ns}} \rightarrow \mathcal{C}^{\text{ns}}$  and  $\text{Hom}(-, 2): \mathcal{BoolAlg} \rightarrow \mathcal{C}^{\text{ns}}$ .)

If one is interested in *relational* structure as well as *operations* on the values of a functor  $V$ , one should look not only at the morphisms among the  $V^n$  but also at their subfunctors. If  $V$  is the covariant (contravariant) representable functor determined by an object  $R$ , an important class of subfunctors of  $V^n$  are those induced by epimorphisms  $R \coprod \dots \coprod R \rightarrow S$  (resp. by subobjects  $S \subseteq R \times \dots \times R$ ).

EXAMPLES. If  $\mathcal{DL}$  is the category of distributive lattices, the functor  $\text{Hom}(-, 2)$ :  $\mathcal{DL} \rightarrow \mathcal{Ens}$  has trivial finitary algebraic structure in the sense of Lawvere, but its finitary representable relational structure is precisely that of partially ordered sets, with " $\leq$ " induced by its graph,  $3 \subseteq 2 \times 2$ . The underlying set functor  $\text{Hom}(1, -)$ :  $\mathcal{Poset} \rightarrow \mathcal{Ens}$  likewise has no operations (so the adjoint tower construction will not get anywhere with it, in contrast to  $\text{Hom}(2, -)$ ), but the relation " $\leq$ " is induced by the epimorphism  $1 \coprod 1 \rightarrow 2$ .

Exercise. The functor  $\text{Hom}(\mathbb{Z}_2, -)$ :  $\mathcal{Group} \rightarrow \mathcal{Ens}$  also has representable relational structure not induced by its operations. What does this say in elementary group-theoretic terms? If you are a group-theorist, find an example.

I mentioned that a monad on  $\mathcal{Ens}$  could correspond to a variety of algebras with a proper *class* of operations, not generated by any *set* of them. An example, noted by Linton [15, p. 90], and studied by Manes [6, pp. 91–118], is the monad  $\mathcal{M}$  arising from the adjunction

$$\begin{array}{ccc} & \text{underlying set} & \\ \mathcal{CptHaus} & \xrightleftharpoons[\text{Stone-Čech}]{\quad} & \mathcal{Ens} \end{array}$$

( $\mathcal{CptHaus}$  = compact Hausdorff spaces). As Manes shows, the lifting functor  $V_2: \mathcal{CptHaus} \rightarrow \mathcal{Ens}^{\mathcal{M}}$  is an equivalence of categories, so compact Hausdorff spaces may be regarded as a variety of (very infinitary) algebras.

A variety of infinitary algebras which does *not* correspond to a monad on  $\mathcal{Ens}$  is that of complete Boolean algebras. For it has been shown [5], [19] that there is no free complete Boolean algebra on countably many generators. This is equivalent to saying that the  $\aleph_0$ -ary complete Boolean operations cannot be indexed by any set.

7. Mirror, mirror.... Let  $\mathcal{Adj}$  denote the category of adjunctions  $\mathcal{P} = (\mathcal{A}, \mathcal{C}; V, F; \eta, \varepsilon)$ . Then the question with which we began §4, “Given an adjunction  $\mathcal{P}$ , if we forget the category  $\mathcal{A}$  what can we ‘remember’ about  $\mathcal{P}$  in terms of  $\mathcal{C}$ ?” is really of the same nature as the question considered in §2. For it asks what “structure” can be put on the values of the forgetful functor:

$$\mathcal{Adj} \xrightarrow{\mathcal{V}_1: \mathcal{P} \mapsto \mathcal{C}} \mathcal{Cat}.$$

The answer turned out to be: a structure of *monad*, giving a factorization:

$$(6) \quad \begin{array}{ccc} & \text{Mon} & \\ & \downarrow & \\ \mathcal{V}_2 & \nearrow & \\ \mathcal{Adj} & \xrightarrow{\mathcal{V}_1} & \mathcal{Cat} \end{array}$$

Of course, the definition of “algebraic structure” must be adjusted to the fact that we are working here with 2-categories, i.e., categories with morphisms between morphisms (e.g., natural transformations between functors, in  $\mathcal{C}\mathcal{at}$ ). Cf. Lawvere [6, pp. 141–155].

Let us think of the objects of  $\mathcal{Mon}$  as 4-tuples,  $\mathcal{M} = (\mathcal{C}; M, \eta, \varepsilon)$ . The construction from a monad  $\mathcal{M}$  of the category of algebras  $\mathcal{C}^M$ , and thence of the adjunction (4), is actually a right adjoint to the functor  $\mathcal{V}_2$  of (6). One would expect, rather, a left adjoint here. This also exists; it is called the Kleisli construction; the new category involved is written  $\mathcal{C}_M$  [2, §VI.5], e.g., if  $\mathcal{M}$  is a monad on  $\mathcal{C}^M$ , so that  $\mathcal{C}^M$  is a variety of algebras,  $\mathcal{C}_M$  turns out to be equivalent to the full subcategory of free algebras in  $\mathcal{C}^M$ , that is, to the theory of this variety.

Both of these adjoints to  $\mathcal{V}_2$  are left inverses to it as well; so it does not appear that (6) will also show higher structure.

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“You sounded like a neophyte who at the age of 30 has just discovered sex, and is so enthusiastic he doesn’t know where to begin! You should have made that example at the end [structure on  $\text{Hom}(\mathbb{Z}_2, —)$ ] the whole talk...”

which I have followed to a large extent in this write-up.

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